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AN INVERSE PROBLEM FOR THE THREE DIMENSIONAL VECTOR HELMHOLTZ EQUATION FOR A PERFECTLY CONDUCTING OBSTACLE

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Abstract—A numerical algorithm for a three-dimensional inverse electromagnetic scattering problem is considered. For time-harmonic waves the Maxwell's equations are reduced to the vector Helmholtz equation. From the knowledge of several far fields generated by an obstacle D when hit by incoming linearly polarized plane waves the boundary ∂D of the obstacle is reconstructed. The obstacle D is supposed to be bounded, connected, with smooth boundary and perfectly conducting. The reconstruction procedure proposed here generalizes the "Herglotz function method" introduced by Colton and Monk [1] for the acoustic problem and is effective in the so-called resonance region.

1. INTRODUCTION

Let $\mathbf{R}^3$ be the three-dimensional real Euclidean space, $\underline{x} = (x, y, z)^T \in \mathbf{R}^3$ be a generic vector, where the superscript T means transpose, $(\cdot,\cdot)$ will denote the Euclidean scalar product and $\|\cdot\|$ the Euclidean norm. In the following we will also use complex vectors, using the same notation.

Let $D \subset \mathbf{R}^3$ be a bounded simply connected domain with smooth boundary ∂D that contains the origin. Let $\underline{E}^i(\underline{x})$ be the electric field associated to a linearly polarized time-harmonic incoming wave propagating in a homogeneous isotropic medium, that is:

$$\underline{E}^i(\underline{x}) = \underline{w}\, e^{ik(\underline{x},\underline{\alpha})}, \tag{1.1}$$

where $\underline{w}, \underline{\alpha} \in \mathbf{R}^3$ with $\|\underline{\alpha}\| = 1$ are given and $k > 0$ is the wave number, moreover, we assume that:

$$\operatorname{div} \underline{E}^i(\underline{x}) = ik\,(\underline{w},\underline{\alpha})\, e^{ik(\underline{x},\underline{\alpha})} = 0, \tag{1.2}$$

where $\underline{E}^i(\underline{x}) = (E^i_x(\underline{x}), E^i_y(\underline{x}), E^i_z(\underline{x}))^T$ and $\operatorname{div} \underline{E}^i(\underline{x}) = \frac{\partial E^i_x(\underline{x})}{\partial x} + \frac{\partial E^i_y(\underline{x})}{\partial y} + \frac{\partial E^i_z(\underline{x})}{\partial z}$. We note that $\underline{w}$ is the polarization vector and $\underline{\alpha}$ is the propagation direction of the incomig electric field. We note that the magnetic field $\underline{H}^i(\underline{x})$ associated to this incoming wave is given by

$$\underline{H}^i(\underline{x}) = \frac{1}{ik} \operatorname{curl} \underline{E}^i(\underline{x}), \tag{1.3}$$

where $\operatorname{curl} \underline{E}^i(\underline{x}) = \left(\frac{\partial E^i_z(\underline{x})}{\partial y} - \frac{\partial E^i_y(\underline{x})}{\partial z}, \frac{\partial E^i_x(\underline{x})}{\partial z} - \frac{\partial E^i_z(\underline{x})}{\partial x}, \frac{\partial E^i_y(\underline{x})}{\partial x} - \frac{\partial E^i_x(\underline{x})}{\partial y} \right)^T$.

Let us denote with $\underline{E}^s(\underline{x})$ the electric field scattered by the obstacle D when hit by the incoming wave $\underline{E}^i(\underline{x})$ and with

$$\underline{E}(\underline{x}) = \underline{E}^i(\underline{x}) + \underline{E}^s(\underline{x}) \tag{1.4}$$

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the total electric field. It is easy to see [2, Chapter 4] that the time-harmonic Maxwell's equations in an homogeneous isotropic medium in $\mathbf{R}^3$ reduce to the vector Helmholtz equation, so that the scattered field $\underline{E}^s(\underline{x})$ satisfies:

$$\Delta \underline{E}^s(\underline{x}) + k^2 \underline{E}^s(\underline{x}) = \underline{0} \quad \text{in} \quad \mathbf{R}^3 \setminus D, \tag{1.5}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ together with the equation:

$$\operatorname{div} \underline{E}^s(\underline{x}) = 0 \quad \text{in} \quad \mathbf{R}^3 \setminus D, \tag{1.6}$$

while the Silver-Müller radiation condition at infinity [2, Paragraph 4.2] reduces to:

$$\operatorname{curl} \underline{E}^s(\underline{x}) \times \hat{\underline{x}} - ik\, \underline{E}^s(\underline{x}) = o\left(\frac{1}{\|\underline{x}\|}\right), \qquad \|\underline{x}\| \to \infty, \tag{1.7}$$

where $\hat{\underline{x}} = \underline{x}/\|\underline{x}\|$, $\underline{x} \neq 0$ and $\times$ is the vector product. Let $\underline{\nu}(\underline{x})$ be the exterior unit normal to ∂D, for a perfectly conducting obstacle D, we will assume the following boundary condition:

$$\underline{E}(\underline{x}) \times \underline{\nu}(\underline{x}) = \underline{0}, \quad \underline{x} \in \partial D. \tag{1.8}$$

In [2] it is shown that $\underline{E}^s(\underline{x})$, solution of the boundary value problem (1.5)–(1.8) has the following expansion:

$$\underline{E}^s(\underline{x}) = \frac{e^{ik\|\underline{x}\|}}{\|\underline{x}\|} \underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}, \underline{w}) + O\left(\frac{1}{\|\underline{x}\|^2}\right), \; \|\underline{x}\| \to \infty, \tag{1.9}$$

where $\underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}, \underline{w})$ is the (electric) far field pattern generated by the incoming wave (1.1) that hits the obstacle D.

In this paper, we introduce a numerical method for an inverse problem for the three-dimensional vector Helmholtz equation (1.5). That is, from the knowledge of the nature of the obstacle (i.e., the fact that the obstacle is perfectly conducting) and from the (electric) far fields $\underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}, \underline{w}_{\underline{\alpha}})$ generated by several (known) incoming waves, we want to recover the shape of the obstacle ∂D.

The inverse acoustic and electromagnetic scattering problems have received a lot of attention in the scientific and technical literature, here we will refer only to the work of Colton and Monk [1], further generalized by the authors [3–5], since the work presented in this paper is inspired by the same ideas. This technique is based on the Herglotz wave function and is supposed to be particularly effective in the resonance region, that is, when

$$kL \cong 1, \tag{1.10}$$

where L is a characteristic length of the obstacle D.

To be more precise, let λ_n, $n = 1, 2, \dots$, be the eigenvalues of the "vector" Laplace operator restricted to the divergence free vector fields with the homogeneous boundary condition (1.8) in the interior of D. Moreover, let $B = \{\underline{x} \in \mathbf{R}^3 \mid \|\underline{x}\| < 1\}$ and ∂B be the boundary of B. We will consider the following inverse problem:

PROBLEM 1.1. Let $\Omega_1 \subseteq \partial B$, $\Omega_2 \subset \mathbf{R}^3$, $\Omega_3 \subset \mathbf{R}$ be three given sets such that $\lambda_i \notin \Omega_3$, $i = 1, 2, \dots$ and let $\underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}, \underline{w}_{\underline{\alpha}})$ be the (electric) far field defined in (1.9). From the knowledge of $\underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}, \underline{w}_{\underline{\alpha}})$ for $\underline{\alpha} \in \Omega_1$, $\underline{w}_{\underline{\alpha}} \in \Omega_2$, $-k^2 \in \Omega_3$ determine the boundary of the obstacle ∂D.

We note that the condition $\lambda_i \notin \Omega_3$, $i = 1, 2, \dots$, is a non-resonance condition, that Ω_1 is the set of the directions of the incoming waves and that Ω_2 is the set of the polarization vectors of the incomig waves.

In this paper, we present a numerical method to solve Problem 1.1, in particular in Section 2, we derive the mathematical relations needed to develop our reconstruction procedure. In Section 3, we present our reconstruction procedure and, finally in Section 4, some numerical experience is shown.

2. THE MATHEMATICAL FORMULATION OF THE INVERSE PROBLEM

For $\underline{x}, \underline{y} \in \mathbf{R}^3$ let

$$\Phi(k\|\underline{x}-\underline{y}\|) = \frac{e^{ik\|\underline{x}-\underline{y}\|}}{4\pi\|\underline{x}-\underline{y}\|} \tag{2.1}$$

be the Green's function of the (scalar) Helmholtz operator with the Sommerfeld radiation condition at infinity.

From Theorems 4.1 and 4.5 of [2] and the fact that $\underline{E}^i(\underline{x})$ is a solution of Equations (1.5) and (1.6), for every $\underline{x} \in \mathbf{R}^3$, it is easy to obtain the following generalization of the well-known Helmholtz formula [6]:

$$\underline{E}^i(\underline{x}) + \operatorname{curl}\int_{\partial D}(\underline{\nu}(\underline{y}) \times \underline{E}(\underline{y}))\,\Phi(k\|\underline{x}-\underline{y}\|)\,d\sigma(\underline{y})$$
$$-\frac{1}{ik}\operatorname{curl}\operatorname{curl}\int_{\partial D}(\underline{\nu}(\underline{y}) \times \underline{H}(\underline{y}))\,\Phi(k\|\underline{x}-\underline{y}\|)\,d\sigma(\underline{y}) = \begin{cases} 0, & \underline{x} \in D \\ \underline{E}(\underline{x}), & \underline{x} \in \mathbf{R}^3 \setminus \overline{D} \end{cases}, \tag{2.2}$$

where $d\sigma(\underline{y})$ is the surface measure on ∂D and

$$\underline{H}(\underline{x}) = \frac{1}{ik}\left[\operatorname{curl}\underline{E}^i(\underline{x}) + \operatorname{curl}\underline{E}^s(\underline{x})\right] \tag{2.3}$$

is the total magnetic field. A formula analogous to (2.2) can be obtained for the magnetic field $\underline{H}(\underline{x})$. From (2.2) and (1.8), in the case of the perfectly conducting obstacle D, we have:

$$\underline{E}^s(\underline{x}) = -\frac{1}{ik}\operatorname{curl}\operatorname{curl}\int_{\partial D}(\underline{\nu}(\underline{y}) \times \underline{H}(\underline{y}))\,\Phi(k\|\underline{x}-\underline{y}\|)\,d\sigma(\underline{y}), \quad \underline{x} \in \mathbf{R}^3 \setminus \overline{D}, \tag{2.4}$$

so that, when $\|\underline{x}\| \to \infty$, we have:

$$\underline{E}^s(\underline{x}) = \frac{e^{ik\|\underline{x}\|}}{\|\underline{x}\|}\frac{ik}{4\pi}\int_{\partial D} e^{-ik(\hat{\underline{x}},\underline{y})}\left\{\underline{\nu}(\underline{y}) \times \underline{H}(\underline{y}) - \hat{\underline{x}}(\hat{\underline{x}}, \underline{\nu}(\underline{y}) \times \underline{H}(\underline{y}))\right\} d\sigma(\underline{y}) + o\left(\frac{1}{\|\underline{x}\|}\right). \tag{2.5}$$

From (1.9) and (2.5), we obtain the following formula for the (electric) far field $\underline{E}_0(\hat{\underline{x}})$:

$$\underline{E}_0(\hat{\underline{x}}) = \frac{ik}{4\pi}\int_{\partial D} e^{-ik(\hat{\underline{x}},\underline{y})}\left\{\underline{\nu}(\underline{y}) \times \underline{H}(\underline{y}) - \hat{\underline{x}}(\hat{\underline{x}}, \underline{\nu}(\underline{y}) \times \underline{H}(\underline{y}))\right\} d\sigma(\underline{y}). \tag{2.6}$$

Let $\underline{g}(\hat{\underline{x}})$ be a square integrable vector valued complex function defined on the surface of the unit sphere ∂B, such that:

$$(\hat{\underline{x}}, \underline{g}(\hat{\underline{x}})) = 0, \ \forall \hat{\underline{x}} \in \partial B \tag{2.7}$$

and let

$$\underline{E}_1(\underline{y}) = \int_{\partial B} \underline{g}(\hat{\underline{x}})\, e^{ik(\hat{\underline{x}},\underline{y})}\, d\lambda(\hat{\underline{x}}), \tag{2.8}$$

where $d\lambda(\hat{\underline{x}})$ is the surface measure on ∂B. It is easy to see that $\underline{E}_1(\underline{y})$ is a divergence free vector field that satisfies the vector Helmholtz equation for any $\underline{y} \in \mathbf{R}^3$. Let $\underline{v} \in \mathbf{R}^3$ be a given vector. We define the vector function

$$\underline{M}(\underline{y}) = -4\pi\left\{\underline{v}\,\overline{\Phi(k\|\underline{x}-\underline{y}\|)} + \frac{1}{k^2}\nabla_{\underline{x}}\left(\underline{v}, \nabla_{\underline{x}}\overline{\Phi(k\|\underline{x}-\underline{y}\|)}\right)\right\}\Bigg|_{\underline{x}=0}, \tag{2.9}$$

where $\nabla_{\underline{x}}$ is the gradient operator with respect to $\underline{x}$, and $\overline{\Phi}$ is the complex conjugate of Φ.

DEFINITION 2.1. *The domain D is said to be a generalized Herglotz domain if the unique solution of the boundary value problem:*

$$(\Delta + k^2)\underline{E}_1(\underline{y}) = \underline{0} \quad \text{in } D, \tag{2.10}$$
$$\operatorname{div}\underline{E}_1(\underline{y}) = 0 \quad \text{in } D, \tag{2.11}$$
$$\underline{\nu} \times \underline{E}_1(\underline{y}) = \underline{\nu} \times \underline{M}(\underline{y}) \quad \text{on } \partial D \tag{2.12}$$

is given by (2.8) for a suitable choice of $\underline{g}(\hat{\underline{x}})$. The function $\underline{g}(\hat{\underline{x}})$ that corresponds to the solution of (2.10)–(2.12) is said to be the generalized Herglotz kernel associated to the domain D.

We note that $\underline{g}(\hat{\underline{x}})$ will depend on $\underline{v}$ and that the class of the generalized Herglotz domains is not empty, since the sphere belongs to it for every $\underline{v}$. Moreover, we remark that the boundary value problem (2.10)–(2.12) has a unique solution due to the non-resonant assumption made on k.

Let us restrict our attention to the class of generalized Herglotz domains and let $\underline{g}(\hat{\underline{x}})$ be a generalized Herglotz kernel, from (2.2), (2.6), (2.8) and (2.12), interchanging the $d\sigma(\underline{y})$ and $d\lambda(\hat{\underline{x}})$ integrations, it is easy to see that:

$$\int_{\partial B} \left(\overline{\underline{g}(\hat{\underline{x}})}, \underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}, \underline{w}_{\underline{\alpha}})\right) d\lambda(\hat{\underline{x}}) = (\underline{w}_{\underline{\alpha}}, \underline{v}) \quad \forall \quad \underline{\alpha} \in \partial B, \underline{w}_{\underline{\alpha}} \in \mathbf{R}^3. \tag{2.13}$$

The inverse Problem 1.1 proposed in Section 1 will be solved in three steps:

(i) from the knowledge of the far fields $\underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}, \underline{w}_{\underline{\alpha}})$ for several $\underline{\alpha}$ and $\underline{w}_{\underline{\alpha}}$ determine, using (2.13), an approximation of the generalized Herglotz kernel $\underline{g}(\hat{\underline{x}})$ of the domain D.
(ii) from $\underline{g}(\hat{\underline{x}})$, using (2.8), determine $\underline{E}_1(\underline{y})$
(iii) from $\underline{E}_1(\underline{y})$, using (2.12), determine ∂D .

3. THE NUMERICAL METHOD

Given $D \subset \mathbf{R}^3$, let $\Omega_1 = \{\underline{\alpha}_i \in \partial B \mid i = 1, 2, \ldots, N\}$ be the set of the directions of the incoming waves, $\Omega_2 = \{\underline{w}_{i,j} \in \mathbf{R}^3 \mid i = 1, 2, \ldots, N,\ j = 1, 2\}$ be the set of the polarization vectors, that is $\underline{w}_{i,1}, \underline{w}_{i,2}$ are the polarization vectors associated to the incoming direction $\underline{\alpha}_i$ $i = 1, 2, \ldots, N$. We note that (1.2) implies:

$$(\underline{\alpha}_i, \underline{w}_{i,j}) = 0 \quad j = 1, 2, \qquad i = 1, 2, \ldots, N. \tag{3.1}$$

From (3.1) and the linearity of our equations, it follows that the choice of two linearly independent vectors $\underline{w}_{i,j}, j = 1, 2$ is sufficient to generate all the information (i.e., (electric) far fields $\underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}_i, \underline{w}_{i,j})$) associated to linearly polarized incoming electromagnetic waves in the direction $\underline{\alpha}_i$. Without loss of generality, we can assume that $(\underline{\alpha}_i, \underline{w}_{i,1}, \underline{w}_{i,2})$ is an orthonormal basis of $\mathbf{R}^3 i = 1, 2, \ldots, N$. Finally, $\Omega_3 = \{-k^2 \mid k > 0\}$ will be chosen as a unique non-resonant value $-k^2$. The data of our inverse problem are the measurements of the (electric) far field $\underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}_i, \underline{w}_{i,j})$, $\forall \hat{\underline{x}} \in \partial B$, $i = 1, 2, \ldots, N$, $j = 1, 2$. In the numerical experience shown in Section 4 these data are obtained solving numerically the "direct" problem (1.5)–(1.8). The direct problem is solved using a T-matrix approach.

Let (θ, ϕ) be the polar angles. We have:

$$\hat{\underline{x}}(\theta, \phi) = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta), \tag{3.2}$$
$$\underline{a}_\theta(\theta, \phi) = (\cos\theta \cos\phi, \cos\theta \sin\phi, -\sin\theta), \tag{3.3}$$
$$\underline{a}_\phi(\theta, \phi) = (-\sin\phi, \cos\phi, 0). \tag{3.4}$$

We note that for each θ and $\phi(\hat{\underline{x}}, \underline{a}_\theta, \underline{a}_\phi)$ is an orthonormal basis of $\mathbf{R}^3$.

Let

$$U_{l,m}(\hat{\underline{x}}) = \gamma_{l,m} P_l^m(\cos\theta) \cos(m\phi), \tag{3.5}$$
$$V_{l,m}(\hat{\underline{x}}) = \gamma_{l,m} P_l^m(\cos\theta) \sin(m\phi), \tag{3.6}$$

be the spherical harmonics, that is P_l^m are the Legendre functions and $\gamma_{l,m}$ are the normalization factors in $L^2(\partial B, d\lambda(\hat{\underline{x}}))$.

We define now the spherical vector harmonics (see [7, p. 1898]):

$$\underline{B}^{0,0}_{l,m}(\hat{\underline{x}}) = \hat{\underline{x}}\, U_{l,m}(\hat{\underline{x}}), \tag{3.7}$$

$$\underline{B}^{1,0}_{l,m}(\hat{\underline{x}}) = \hat{\underline{x}}\, V_{l,m}(\hat{\underline{x}}), \tag{3.8}$$

$$\underline{B}^{0,1}_{l,m}(\hat{\underline{x}}) = (l(l+1))^{-\frac{1}{2}}\, \|\underline{x}\|\, \nabla U_{l,m}(\hat{\underline{x}}), \tag{3.9}$$

$$\underline{B}^{1,1}_{l,m}(\hat{\underline{x}}) = (l(l+1))^{-\frac{1}{2}}\, \|\underline{x}\|\, \nabla V_{l,m}(\hat{\underline{x}}), \tag{3.10}$$

$$\underline{B}^{0,2}_{l,m}(\hat{\underline{x}}) = (l(l+1))^{-\frac{1}{2}}\, \mathrm{curl}\, (\underline{x} U_{l,m}(\hat{\underline{x}})), \tag{3.11}$$

$$\underline{B}^{1,2}_{l,m}(\hat{\underline{x}}) = (l(l+1))^{-\frac{1}{2}}\, \mathrm{curl}\, (\underline{x} V_{l,m}(\hat{\underline{x}})). \tag{3.12}$$

It is well known that the spherical vector harmonics are an orthonormal basis of $L^2(\partial B, \mathbf{R}^3, d\lambda(\hat{\underline{x}}))$.

Our computation proceeds in four steps:

STEP 1. For each (i,j) compute the "'Fourier coefficients" of $\underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}_i, \underline{w}_{i,j})$, $i = 1, 2, \ldots, N$, $j = 1, 2$.

Given $L_{max} > 0$, we assume that the (electric) far field $\underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}_i, \underline{w}_{i,j})$ can be approximated by a truncated Fourier series, that is

$$\underline{E}_0(\hat{\underline{x}}) = \sum_{\tau=1,2} \sum_{s=0,1} \sum_{l=1}^{L_{max}} \sum_{m=s}^{l} f^{s,\tau}_{i,j,l,m}\, \underline{B}^{s,\tau}_{l,m}(\hat{\underline{x}}), \tag{3.13}$$

where the Fourier coefficients $f^{s,\tau}_{i,j,l,m}$ are determined computing the appropriate integrals such as:

$$f^{s,\tau}_{i,j,l,m} = \int_{\partial B} (\underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}_i, \underline{w}_{i,j}), \underline{B}^{s,\tau}_{l,m}(\hat{\underline{x}}))\, d\lambda(\hat{\underline{x}}), \qquad s = 0, 1; \quad \tau = 1, 2. \tag{3.14}$$

STEP 2. From the Fourier coefficients of $\underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}_i, \underline{w}_{i,j})$, $i = 1, 2, \ldots, N$, $j = 1, 2$ to the generalized Herglotz kernel $\underline{g}(\hat{\underline{x}})$.

Given $\underline{v} \in \mathbf{R}^3$, let $\underline{g}(\hat{\underline{x}})$ be the generalized Herglotz kernel associated to the domain D. We assume for $\underline{g}(\hat{\underline{x}})$ the expression of a truncated Fourier expansion, that is:

$$\underline{g}(\hat{\underline{x}}) = \sum_{\tau=1,2} \sum_{s=0,1} \sum_{l=1}^{L_g} \sum_{m=s}^{l} g^{s,\tau}_{l,m}\, \underline{B}^{s,\tau}_{l,m}(\hat{\underline{x}}), \tag{3.15}$$

where $0 < L_g \leq L_{max}$ and the coefficients $g^{s,\tau}_{l,m}$ are complex numbers. From (2.13), using the orthogonality properties of the spherical vector harmonics, we have:

$$\begin{aligned}\int_{\partial B} \left(\overline{\underline{g}(\hat{\underline{x}})}, \underline{E}_0(\hat{\underline{x}}, k, \underline{\alpha}_i, \underline{w}_{i,j})\right) d\lambda(\hat{\underline{x}}) &= \sum_{\tau=1,2} \sum_{s=0,1} \sum_{l=1}^{L_g} \sum_{m=s}^{l} f^{s,\tau}_{i,j,l,m}\, \overline{g^{s,\tau}_{l,m}} \\ &= (\underline{w}_{i,j}, \underline{v}), \qquad i = 1, 2, \ldots, N, \quad j = 1, 2,\end{aligned} \tag{3.16}$$

so that (3.16) can be interpreted as $2N$ linear equations in the unknowns $\{g^{s,\tau}_{l,m}\}$. This linear system has $2L_g(L_g+2)$ unknowns, so that in order to determine them we need $N \geq L_g(L_g+2)$.

If the obstacle D has some symmetry, such as cylindrical symmetry around the z-axis and/or symmetry with respect to the equator, then a similar symmetry can be assumed for $\underline{g}(\hat{\underline{x}})$. This assumption reduces substantially the number of unknowns in (3.16) and, as a consequence, the number $2N$ of incoming waves needed to recover the desired approximation to $\underline{g}(\hat{\underline{x}})$.

STEP 3. From the generalized Herglotz kernel $\underline{g}(\hat{\underline{x}})$ to $\underline{E}_1(\underline{y})$.

From (2.8) and (3.15), we have

$$\begin{aligned}\underline{E}_1(\underline{y}) = 4\pi \sum_{s=0,1} \sum_{l=1}^{L_g} \sum_{m=s}^{l} \Bigg\{ g_{l,m}^{s,1} i^{l-1} \Bigg[(l(l+1))^{\frac{1}{2}} \frac{j_l(k\|\underline{y}\|)}{k\|\underline{y}\|} \underline{B}_{lm}^{s,0}(\hat{\underline{y}}) \\ + \left[\frac{l+1}{k\|\underline{y}\|} j_l(k\|\underline{y}\|) - j_{l+1}(k\|\underline{y}\|)\right] \underline{B}_{l,m}^{s,1}(\hat{\underline{y}}) \Bigg] + g_{l,m}^{s,2} i^l j_l(k\|\underline{y}\|) \underline{B}_{l,m}^{s,2}(\hat{\underline{y}}) \Bigg\},\end{aligned} \tag{3.17}$$

where j_l is the spherical Bessel function of order l.

STEP 4. From $\underline{E}_1(\underline{y})$ to the boundary of the obstacle ∂D.

Let (r, θ, ϕ) be the polar variables, we assume that there exists $0 < a < b < \infty$ and a function $f(\theta, \phi)$ with $a < f < b$ such that $\partial D = \{r = f(\theta, \phi) \mid 0 \leq \theta \leq \pi\ ,\ 0 \leq \phi < 2\pi\}$. We approximate $f(\theta, \phi)$ with a truncated Fourier series, that is:

$$f(\theta, \phi) = \sum_{l=0}^{L_\rho} \sum_{m=0}^{l} c_{l,m,1} U_{l,m}(\hat{\underline{x}}) + \sum_{l=1}^{L_\rho} \sum_{m=1}^{l} c_{l,m,2} V_{l,m}(\hat{\underline{x}}), \tag{3.18}$$

where $L_\rho \geq 0$ is chosen, depending on ∂D, i.e., simple obstacles can be reconstructed with $L_\rho = 4, 6$. Moreover, if ∂D has some symmetries, these symmetries can be translated into properties of $\{c_{l,m,1}, c_{l,m,2}\}$. Let $\underline{c} = \{c_{l,m,1}, c_{l,m,2}\}$ $0 \leq l \leq L_\rho, 0 \leq m \leq l$ be an $(L_\rho + 1)^2$ dimensional vector. The unknown boundary ∂D is obtained by minimizing, with respect to $\underline{c}$, the function

$$I_1(\underline{c}) = \left\{ \int_0^{2\pi} d\phi \int_0^\pi \left| \|(\underline{E}_1 - \underline{M}) \times \underline{\nu}\| \right|^2 \sin\theta \, d\theta \right\}^{\frac{1}{2}}, \tag{3.19}$$

where the functions $\underline{E}_1, \underline{M}, \underline{\nu}$ in (3.19) are computed in $(r = f(\theta, \phi), \theta, \phi)$. The integrals in (3.19) are approximated by some elementary quadrature formulae.

When the minimization of the function $I_1(\underline{c})$ does not give a satisfactory reconstruction of ∂D, we minimize

$$P(\underline{c}) = I_1(\underline{c}) + I_2(\underline{c}), \tag{3.20}$$

where $I_2(\underline{c})$ is a penalization term.

For a large class of smooth surfaces including the ones with cylindrical symmetry with respect to the z-axis, we have $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \phi} = 0$ at $\theta = 0$ (North pole) and $\theta = \pi$ (South pole), moreover, if D is also symmetric with respect to the equator, we have $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \phi} = 0$ at $\theta = \frac{\pi}{2}$ (the Equator).

When $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \phi} = 0$ the relation (2.12) becomes a nonlinear equation for f that can be solved, for example, using the bisection method. In this way we obtain f_1^*, f_2^*, f_3^* estimated values of $f(\theta, \phi)$, when $\theta = \theta_1, \theta_2, \theta_3$ with $\theta_1 = 0, \theta_2 = \pi, \theta_3 = \frac{\pi}{2}$. The penalization term $I_2(\underline{c})$ is given by

$$I_2(\underline{c}) = \sum_{i=1}^{3} p_i (f(\theta_i, \phi) - f_i^*)^2, \tag{3.21}$$

where $f(\theta, \phi)$ is given by (3.18) and $p_i \geq 0, i = 1, 2, 3$ are weight factors. The minimization of the functions $I_1(\underline{c})$ and $P(\underline{c})$ are performed starting from the unit sphere as an initial guess using a quasi-Newton algorithm of the IMSL Software library [8] or the stochastic global minimization algorithm introduced in [9, 10].

4. THE NUMERICAL EXPERIENCE

The surfaces ∂D considered are the following ones:

(1) Oblate Ellipsoid $$\left(\frac{2}{3}x\right)^2 + \left(\frac{2}{3}y\right)^2 + z^2 = 1 \tag{4.1}$$

(2) Prolate Ellipsoid $$x^2 + y^2 + \left(\frac{2}{3}z\right)^2 = 1 \tag{4.2}$$

(3) Vogel's Peanut $$r = \frac{2}{3}\left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right)^{\frac{1}{2}} \tag{4.3}$$

(4) Vertical Peanut $$r = 1 + \frac{1}{2}\cos 2\theta \tag{4.4}$$

(5) Horizontal Platelet $$r = 1 - \frac{1}{2}\cos 2\theta \tag{4.5}$$

(6) Reverse Platelet $$r = \frac{5}{4} + \frac{1}{4}\cos 4\theta \tag{4.6}$$

(7) Short Cylinder $$\left(\left(\frac{2}{3}x\right)^2 + \left(\frac{2}{3}y\right)^2\right)^5 + z^{10} = 1 \tag{4.7}$$

(8) Pseudo Apollo $$r = \frac{3}{5}\left(\frac{17}{4} + 2\cos 3\theta\right)^{\frac{1}{2}} \tag{4.8}$$

All these surfaces are cylindrically symmetric with respect to the z-axis and the surfaces 1, 2, 3, 4, 5, 6 and 7 are also symmetric with respect to the equator. These symmetries are always exploited to reduce the number of Fourier coefficients in the expansions of the generalized Herglotz kernels and of the surfaces $f(\theta, \phi)$.

We observe that the obstacles D, corresponding to 1, 2 and 7, are convex and the ones corresponding to 3, 4, 5, 6 and 8 are not convex. Finally a characteristic length L of the obstacles can be chosen equal to 1.

The directions and polarizations of the incoming waves are defined by

$$\Omega_1 = \left\{\underline{\alpha}_i = \underline{\hat{x}}(\theta_i, 0), \theta_i = \frac{\pi}{L_g+1}i, \qquad i = 0, 1, \ldots, (L_g+1)\right\}, \tag{4.9}$$

$$\begin{aligned}\Omega_2 = \Bigg\{\underline{w}_{i,j} \,\Bigg|\, &(\underline{\alpha}_i, \underline{w}_{i,1}, \underline{w}_{i,2}) \text{ is an orthonormal basis of } \mathbf{R}^3, \\ &i = 0, 1, \ldots, (L_g+1), j = 1, 2\Bigg\}.\end{aligned} \tag{4.10}$$

The values of k considered are 2.5, 3 or 4, so that the product kL is of order one, that is we are working in the resonance region. In our reconstructions, we use only the penalization term on the equator as declared in Table 4.1; the penalization terms at the poles are never used. Moreover, we always use $L_{max} = L_g$, and the data , that is the Fourier coefficients of the far fields $f^{s,T}_{i,j,l,m}$, obtained from Step 1, are substituted with

$$(1 + \epsilon\zeta) f^{s,T}_{i,j,l,m}, \tag{4.11}$$

where $\epsilon > 0$ is a parameter and ζ is a uniformly distributed random number in [-1,1].

For $j = 0, 1, \ldots, 36$, let $\theta_j = \pi j/36$, $f(\theta_j, 0)$ be the exact values of the surface given by (4.1)–(4.8) and $f_c(\theta_j, 0)$ be the values reconstructed performing the numerical procedure described in Section 3. The relative L^2 error in the points $\{(\theta_j, 0) \mid j = 0, 1, \ldots, 36\}$, that is:

$$E_{L^2} = \left[\frac{\sum\limits_{j=0}^{36}(f(\theta_j, 0) - f_c(\theta_j, 0))^2}{\sum\limits_{j=0}^{36} f^2(\theta_j, 0)}\right]^{\frac{1}{2}} \tag{4.12}$$

Table 4.1 The numerical results.

Object	Reconstruction	L_{max}	L_p	k	ϵ	Equator penalization term	E_{L^2}
Oblate Ellipsoid	1	7	4	3	0	no	0.0047
”	2	7	4	3	0.01	no	0.0093
”	3	7	4	3	0.05	no	0.0304
”	4	7	4	3	0.10	no	0.0510
Prolate Ellipsoid	5	7	4	3	0	no	0.0323
Vogel's Peanut	6	7	4	3	0	no	0.0067
”	7	7	4	3	0.01	no	0.0264
”	8	7	4	3	0.025	no	0.0120
”	9	7	4	3	0.05	no	0.1551
Vertical Peanut	10	9	4	4	0	yes	0.0364
Horizontal Platelet	11	7	4	3	0	no	0.0956
Reverse Platelet	12	9	4	4	0	yes	0.0315
Short Cylinder	13	9	6	2.5	0	yes	0.0521
Pseudo Apollo	14	9	4	4	0	no	0.0559

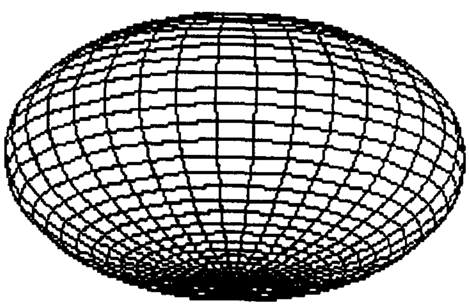

Figure 4.1a. Original.

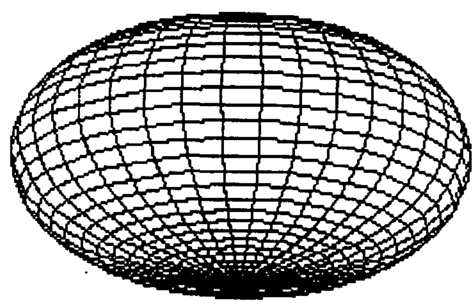

Figure 4.1b. Reconstruction 1.

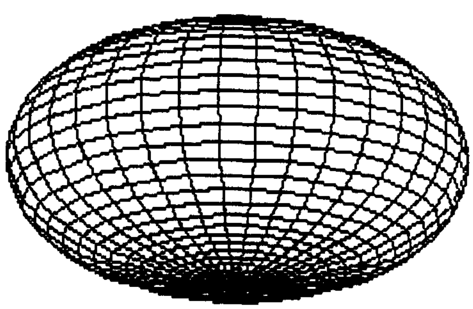

Figure 4.1c. Reconstruction 2.

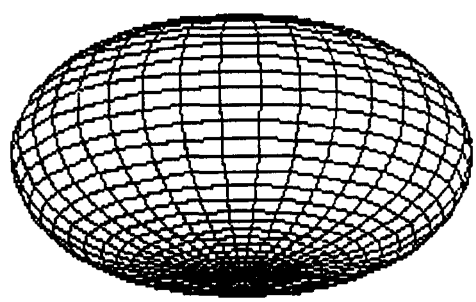

Figure 4.1d. Reconstruction 3.

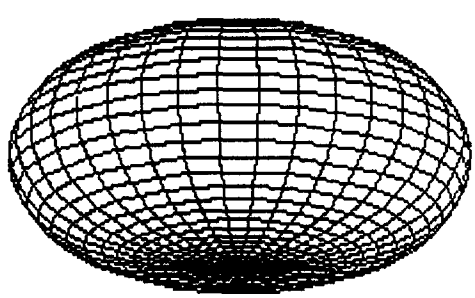

Figure 4.1e. Reconstruction 4.

is used as a performance index. The results obtained are shown in Table 4.1, Figures 4.1a–4.1e, Figures 4.2a–4.2e and Figures 4.3a–4.3f.

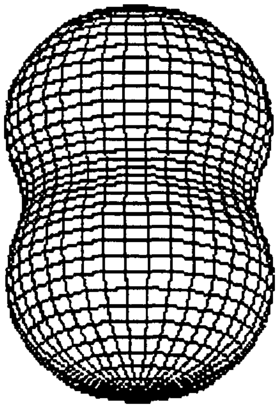

Figure 4.2a. Original.

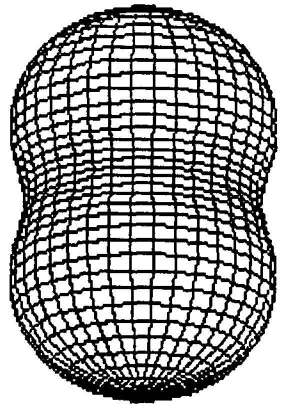

Figure 4.2b. Reconstruction 6.

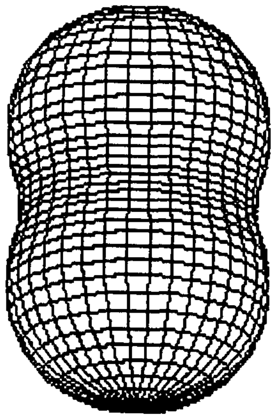

Figure 4.2c. Reconstruction 7.

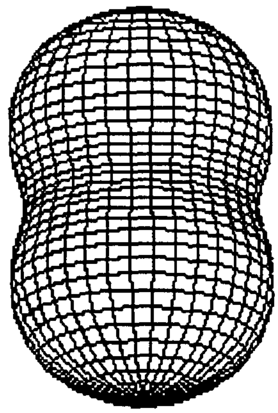

Figure 4.2d. Reconstruction 8.

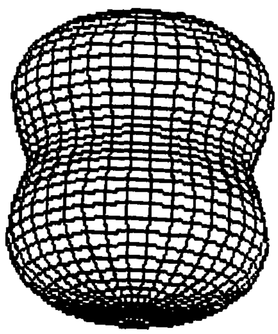

Figure 4.2e. Reconstruction 9.

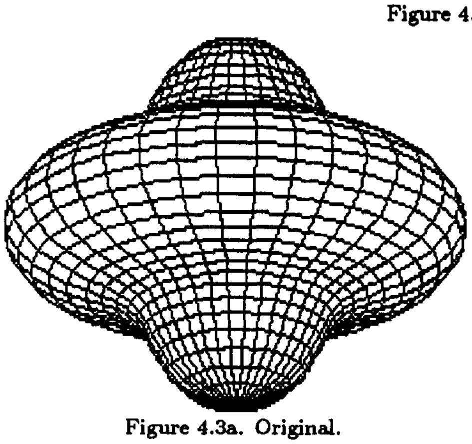

Figure 4.3a. Original.

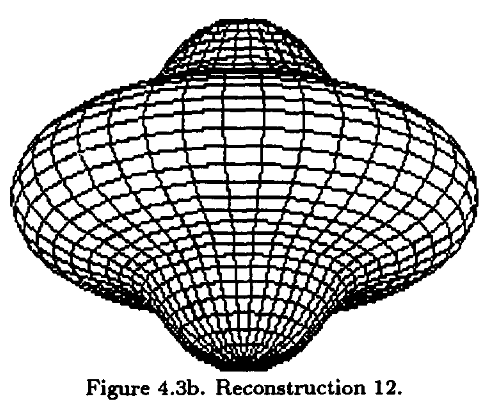

Figure 4.3b. Reconstruction 12.

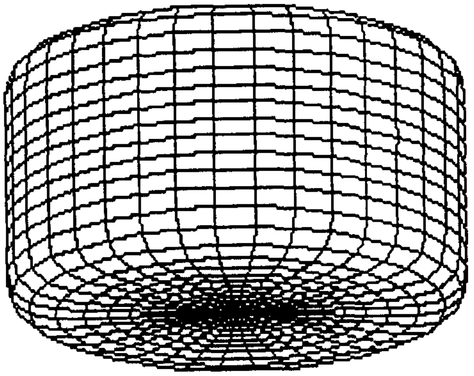

Figure 4.3c. Original.

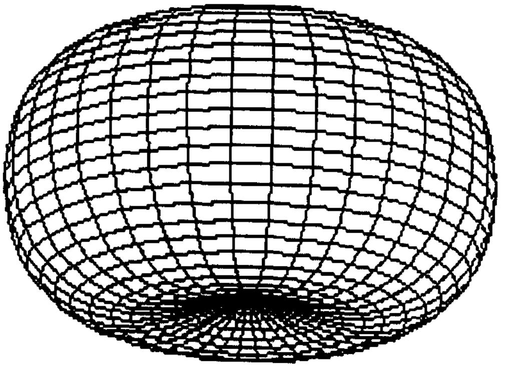

Figure 4.3d. Reconstruction 13.

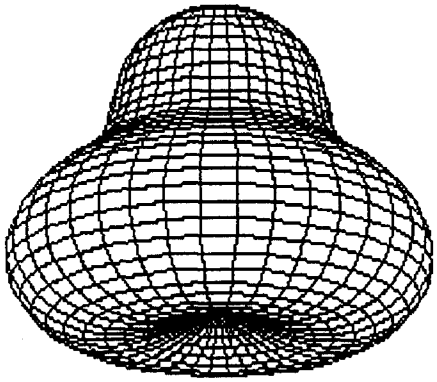

Figure 4.3e. Original.

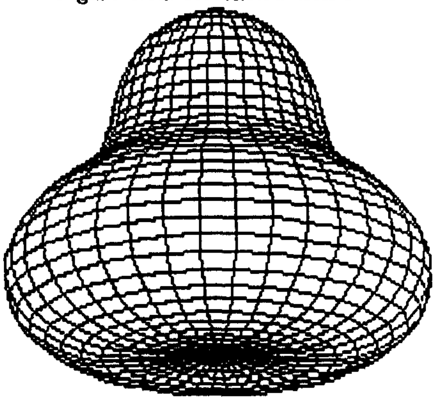

Figure 4.3f. Reconstruction 14.

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